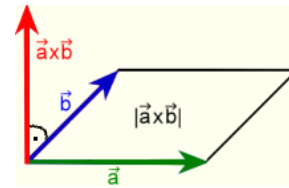


## Q11 \* Mathematik \* Vektorprodukt (Kreuzprodukt)

$$\begin{array}{r}
 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
 \begin{array}{l}
 \text{1. Zeile: } a_2 b_3 - a_3 b_2 \\
 \text{2. Zeile: } a_3 b_1 - a_1 b_3 \\
 \text{3. Zeile: } a_1 b_2 - a_2 b_1
 \end{array}
 \end{array}$$



Vektorprodukt zweier Vektoren:  $\vec{a} \times \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 \cdot b_3 - a_3 \cdot b_2 \\ a_3 \cdot b_1 - a_1 \cdot b_3 \\ a_1 \cdot b_2 - a_2 \cdot b_1 \end{pmatrix}$

Satz:  $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin \varphi$  wobei  $\varphi$  der Winkel zwischen  $\vec{a}$  und  $\vec{b}$  ist.

Beweis:

$$\begin{aligned}
 (\vec{a} \times \vec{b})^2 &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
 &= (a_2 b_3)^2 - 2 a_2 a_3 b_2 b_3 + (a_3 b_2)^2 + (a_3 b_1)^2 - 2 a_3 a_1 b_1 b_3 + (a_1 b_3)^2 \\
 &\quad + (a_1 b_2)^2 - 2 a_1 a_2 b_1 b_2 + (a_2 b_1)^2 \\
 &= (a_1 b_2)^2 + (a_1 b_3)^2 + (a_2 b_1)^2 + (a_2 b_3)^2 + (a_3 b_1)^2 + (a_3 b_2)^2 \\
 &\quad - 2 a_1 a_2 b_1 b_2 - 2 a_1 a_3 b_1 b_3 - 2 a_2 a_3 b_2 b_3
 \end{aligned}$$

$$|\vec{a}|^2 \cdot |\vec{b}|^2 \cdot (\sin \varphi)^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot (1 - (\cos \varphi)^2) = |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot \left(1 - \frac{(\vec{a} \cdot \vec{b})^2}{|\vec{a}|^2 |\vec{b}|^2}\right)$$

$$\begin{aligned}
 &= |\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = \\
 &= (a_1^2 + a_2^2 + a_3^2) \cdot (b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
 &= (a_1 b_1)^2 + (a_1 b_2)^2 + (a_1 b_3)^2 + (a_2 b_1)^2 + (a_2 b_2)^2 + (a_2 b_3)^2 \\
 &\quad + (a_3 b_1)^2 + (a_3 b_2)^2 + (a_3 b_3)^2 \\
 &\quad - \left( (a_1 b_1)^2 + a_1 b_1 a_2 b_2 + a_1 b_1 a_3 b_3 + a_2 b_2 a_1 b_1 + (a_2 b_2)^2 + a_2 b_2 a_3 b_3 \right. \\
 &\quad \left. + a_3 b_3 a_1 b_1 + a_3 b_3 a_2 b_2 + (a_3 b_3)^2 \right) \\
 &= (a_1 b_2)^2 + (a_1 b_3)^2 + (a_2 b_1)^2 + (a_2 b_3)^2 + (a_3 b_1)^2 + (a_3 b_2)^2 \\
 &\quad - 2 a_1 a_2 b_1 b_2 - 2 a_1 a_3 b_1 b_3 - 2 a_2 a_3 b_2 b_3
 \end{aligned}$$

Also  $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot (\sin \varphi)^2 \Rightarrow$

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot |\sin \varphi| = |\vec{a}| \cdot |\vec{b}| \cdot \sin \varphi$$